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INFINITE DETERMINANTS IN THE THEORY OF MATHIEU'S AND HILL'S EQUATIONS

by

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Abstract

The existence of even or odd solutions of Mathieu's equation with period π or 2π depends on the vanishing of certain infinite determinants. It is shown that these determinants can be expressed in terms of the values of the even or odd solutions at $x = \pi/2$, where x is the independent variable. The corresponding result for Hill's determinant (which can be expressed in terms of the values of certain solutions at $x = \pi$) has long since been found. The method applied in the present report uses theorems about the order of growth of the solutions both with respect to x and the parameters. A representation of the solutions as a finite Fourier integral can also be derived by this method. We indicate how far the results can be extended to the case of more general equations of Hill's type. The periodic solutions of the inhomogeneous Mathieu equations are constructed by using an inversion formula for certain infinite matrices. The connection between Hill's equation and the construction of transparent layers is reviewed.

Introduction We shall write Mathieu's equation in the form

$$(1) \qquad y'' + 4(\omega^2 + 2t\cos 2x)y = 0,$$

where y'' denotes d^2y/dx^2. The most widely treated problem in the theory of this equation is the following one: Let t be real. Find the values of ω for which there exists a solution $y(x)$ of (1) such that either

$$(2) \qquad y(x+\pi) = y(x)$$

or

$$(3) \qquad y(x+2\pi) = y(x), \quad (y(x+\pi) \neq y(x)).$$

One method of finding a solution of (1) satisfying (2) or (3) is to satisfy (1) by an infinite series

$$(4) \qquad y = \sum_{n=-\infty}^{\infty} c_r \exp(i2nx),$$

or by a corresponding series if (3) is to be satisfied. By substituting (4) into (1) we find a homogeneous system of infinitely many linear equations for the c_r. The determinant of this system - after an appropriate normalization of the coefficients in the main diagonal - is Hill's determinant, which is defined explicitly in Section II and is denoted by $\Delta_o(\omega,t)$. The vanishing of $\sin^2\pi\omega\,\Delta_o$ is a necessary and sufficient condition for the existence of a solution of period π.

Another method of dealing with the problem of periodic solution is based on Floquet's theorem. Using the fact that the coefficient of y in (1) is an even continuous function of x with period π, one arrives at the following result:

Let $y_1(x)$ be the solution of (1) defined by $y_1(0) = 1$, $y_1'(0) = 0$ and let $y_2'(x)$ be the solution defined by $y_2(0) = 0$, $y_2'(0) = 1$. Then (1) has a solution satisfying (2) if and only if

$$(5) \qquad y_1(\pi) = 1$$

and a solution satisfying (3) if and only if

$$(6) \qquad y_1(\pi) = -1.$$

The connection with the methods based on Hill's determinant is established by the well-known relation

$$y_1(\pi) = 1 - 2\,\Delta_0 \sin^2 \pi\omega . \tag{7}$$

Recently Schaefke [10] has stressed the point that for any differential equation

$$y'' + q(x)y = 0, \tag{8}$$

in which $q(x)$ is continuous for all x and

$$q(x) = q(-x), \quad q(x+\pi) = q(x) , \tag{9}$$

the following elementary result holds: Let $y_1(x)$ and $y_2(x)$ be the normalized solutions of (8) which are defined as above. Then (8) has

$$\begin{array}{lll} \text{an even solution of period } \pi & \text{if and only if} & y_1'(\frac{\pi}{2}) = 0 \\ \text{an odd solution of period } \pi & \text{"} & y_2(\frac{\pi}{2}) = 0 \\ \text{an even solution of period } 2\pi & \text{"} & y_1(\frac{\pi}{2}) = 0 \\ \text{an odd solution of period } 2\pi & \text{"} & y_2'(\frac{\pi}{2}) = 0 . \end{array} \tag{10}$$

The statements in (10) express the periodic boundary conditions in terms of an ordinary boundary value problem for the interval $(0, \pi/2)$. Also, they show that the conditions expressed by (2) or (3) can be split into two independent conditions. This becomes obvious from the formulas

$$\begin{aligned} y_1(\pi) - 1 &= y_1'(\tfrac{\pi}{2})\, y_2(\tfrac{\pi}{2}) \\ y_1(\pi) + 1 &= y_1(\tfrac{\pi}{2})\, y_2'(\tfrac{\pi}{2}) \end{aligned} \tag{11}$$

(which, again, are of an elementary nature).

In the case of Mathieu's equation (1), the investigation of the four possibilities listed under (10) is a standard feature of the theory. Frequently, the problem of finding, for example, an even solution of period π, has been treated by using infinite determinants. If such a solution exists it can be expanded in a series

$$y(x) = \sum_{r=0}^{\infty} g_r \cos 2r\, x \tag{12}$$

On substituting (12) into (1) we find again that a certain infinite determinant must vanish. We write the resulting condition in the form

(13) $$-2\omega \sin \pi\omega \, C_+ = 0 ,$$

where C_+ is an infinite determinant which is given explicitly in equation (20) of Section II. Corresponding to the three other cases listed in (10), three more infinite determinants are defined in Section II; they are denoted by S_+, C_-, and S_- .

This is the point at which the investigations of this report stand. It is proved directly, from an inspection of the infinite determinants C_+, S_+, that

(14) $$\begin{cases} (\sin^2 \pi \omega) \Delta_0 = (2\omega \sin \pi\omega) \, C_+ \dfrac{\sin \pi \omega}{2\omega} \, S_+ \\ (\sin^2 \pi \omega) \Delta_0 - 1 = (\cos^2 \pi \omega) \, C_- \, S_- \end{cases}$$

(see theorem 2 in Section II). As a consequence, four analogues of (7) can be proved; one of these, for instance, is

(15) $$-(2\omega \sin \pi \omega) \, C_+ = y_1'(\tfrac{\pi}{2}) .$$

(theorem 3, Section IV).

Another result which can be obtained from the particular nature of the determinants C_+, . . . , is an approach to the inhomogeneous Mathieu equation whose right-hand side has a period π . It is shown in Section I how the method of infinitely many linear equations can be used for the construction of a periodic solution of the inhomogeneous equation in the case where the homogeneous equation does not have such a solution.

The explicit form of C_+ and the general observations of Section I about the determinants of this type reveal that the left-hand side of (13) is an analytic function of ω which can be written as a sum having the form

$$-2\omega \sin \pi \omega + \sum_{n=-\infty}^{+\infty} s_n \frac{\sin \pi \omega}{\omega + n} .$$

Consequently it is to be expected that $-2\omega \sin \pi\omega (C_+ - 1)$ can be written in the form of a finite Fourier integral, since

$$\frac{\sin \pi\omega}{\omega + n} = (-1)^n \int_{-\pi/2}^{\pi/2} e^{2i\omega\phi} e^{2in\phi} d\phi \quad .$$

Because of (15) one may expect that in general, for any value of x and any solution of $y(x)$ of (1) or its first derivative, a similar representation will be possible if certain initial terms are first subtracted from $y(x)$. That this is indeed the case is stated in Section V, theorem 4. This theorem may be described as a Fourier representation of the solutions of (1) with respect to the parameter ω.

Finally, in Section II we prove the well-known theorem about the mutually exclusive nature of the four conditions (10), using the infinite determinants C_+, S_+, ... This theorem is the negative aspect (in the case of Mathieu's equation) of the problem of finding equations of Hill's type (8) which have two linearly independent solutions of the same period. In section VII, the connection between this problem and the problem of constructing "transparent" media is summarized and reviewed.

Section VI indicates the possibilities for generalizing the results. The results obtained for Mathieu's equation could be proved completely for Hill's equation (8) as well if the analogue of (14) could be proved. Without that only some results, including theorem 4, can be generalized.

The methods used in this report are based upon theorems about functions of a given order of growth. This approach to the theory of Hill's equation is due to Schaefke.

I wish to thank Professor J. Keller and Dr. C. J. Bouwkamp for their valuable help in preparing this report. I am indepted to Professor Keller for the method of Section I for the inversion of certain infinite matrices. Dr. Bouwkamp called my attention to several papers on Mathieu functions and Hill's equations and gave me his unpublished results which are mentioned in Section VII.

I. On a Special Class of Infinite Matrices and Determinants

We shall consider matrices of the following type: The elements in the main diagonal are equal to unity, and the only other elements which may be different from zero are those in the diagonals adjacent to the main diagonal. The most general matrix of this type can be indicated by

$$(1)\qquad \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & p_{-3} & 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & q_{-3} & 1 & p_{-2} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & q_{-2} & 1 & p_{-1} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & q_{-1} & 1 & p_o & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & q_o & 1 & p_1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & q_1 & 1 & p_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

We will denote this matrix by ${}^{-\infty}_{+\infty}M(p_\nu,q_\nu)$ and its determinant (if it exists in the sense explained below) by ${}^{-\infty}_{+\infty}D(p_\nu,q_\nu)$; the subscript $+\infty$ and the superscript $-\infty$ denote the range of rows and columns or the range of values of ν which defines the set of parameters p_ν, q_ν.

Similarly, we shall denote the infinite matrix

$$(2)\qquad \begin{matrix} 1 & p_o & 0 & 0 & \cdot \\ q_o & 1 & p_1 & 0 & \cdot \\ 0 & q_1 & 1 & p_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

by ${}^{o}_{\infty}M(p_\nu,q_\nu)$; this means that the matrix is infinite only in the direction of increasing subscripts denoting rows and columns.

If the rows containing $p_{\nu_1}, p_{\nu_2}, p_{\nu_3}$ ... and the columns containing $q_{\mu_1}, q_{\mu_2}, q_{\mu_3}$, ... are taken out of M, the remaining matrix will be denoted by

$$M_{\nu_1,\mu_1;\ \nu_2,\mu_2;\nu_3,\ \mu_3,\ \cdots}$$

and the corresponding notation will be used for determinants and their sub-determinants.

Definition of a determinant of an infinite matrix. Let M be an infinite matrix and let Δ_1, Δ_2, Δ_3, ... be a sequence of finite determinants formed from the elements of M (in the arrangement in which they occur in M). These determinants are such that

(I) The main diagonal of each Δ_ℓ (ℓ = 1,2,3,...) is part of the main diagonal of M

(II) Every element of M appears in a Δ_ℓ and every Δ_ℓ is a subdeterminant of $\Delta_{\ell+1}$, and

(III) $\lim\limits_{\ell \to \infty} \Delta_\ell$ = D exists.

Then, if D is independent of the particular choice of the sequence Δ_ℓ, we shall call D the value of the determinant of M. We have:

Lemma 1. Let

(3) $$w_\nu = p_\nu q_\nu .$$

Then the determinant D of a matrix M of type (1) or (2) exists if

$$\sum |w_\nu|$$

converges, where the sum is taken over all values of ν from $-\infty$ to $+\infty$ or (in the case of a ${}^{0}_{\infty}M$) from 0 to ∞. The value of D depends only on the w_ν and not on the p_ν, c_ν themselves.

We may prove this lemma by observing that a finite subdeterminant, say ${}^{o}_{n}D$, can be shown to be

(4) $$ {}^{o}_{n}D = 1 + \sum_{o}^{n} w_\nu + \sum{}^{*} w_{\nu_1} w_{\nu_2} + \sum_{\nu_1,\nu_2,\nu_3}^{*} w_{\nu_1} w_{\nu_2} w_{\nu_3} + \dots, $$

where the sums are taken over $\nu, \nu_1, \nu_2, \nu_3, \dots$ and where the asterisk implies that all those terms which involve a product $w_\nu w_{\nu+1}$ of two consecutive factors w_ν should be left out of the sum. It follows that the infinite product

(5) $$ \prod_{\nu} (1 + |w_\nu|) $$

majorizes the sum of the absolute values of the terms in (4). Then the convergence of an infinite product of type (5) can be proven, using the well-known convergence test, and this proves lemma 1.

Using the formula for the expansion of a determinant in terms of subdeterminants belonging to the elements of a fixed row, we derive easily the

Recurrence relations:

(6) $$ {}^{-\infty}_{\infty}D = \left({}^{-\infty}_{-1}D\right)\left({}^{2}_{\infty}D\right) - w_o\left({}^{-\infty}_{-2}D\right)\left({}^{2}_{\infty}D\right) - w_1\left({}^{-\infty}_{-1}D\right)\left({}^{3}_{\infty}D\right), $$

(6') $$ {}^{-\infty}_{\infty}D = \left({}^{-\infty}_{-2}D\right)\left({}^{1}_{\infty}D\right) - w_{-1}\left({}^{-\infty}_{-3}D\right)\left({}^{1}_{\infty}D\right) - w_o\left({}^{-\infty}_{-2}D\right)\left({}^{2}_{\infty}D\right), $$

(7) $$ {}^{o}_{\infty}D = {}^{o}_{\infty}D_{1,1} - w_o\, {}^{o}_{\infty}D_{1,1;2,2} \equiv {}^{1}_{\infty}D - w_o\, {}^{2}_{\infty}D, $$

(8) $$ {}^{o}_{\infty}D = \lim_{n\to\infty} z_n $$

where z_n is defined by the recurrence relation

(9) $$ z_{n+1} = z_n - w_n z_{n-1}, \quad (z_{-2} = 0, z_{-1} = 1, z_o = 1, n = 0,1,2,\dots). $$

Here z_{r-1} is actually the determinant whose elements are those elements in ${}^{o}_{\infty}D$ which appear both in the first n rows and in the first n columns.

Now we shall try to find a formal inverse M^{-1} of a matrix $M = {}_{\infty}\overset{o}{M}$ of type (2). For this purpose we shall use a method found by Professor J. Keller. Let Y,X denote the infinite matrices indicated by

$$(10) \qquad \begin{matrix} 1 & 0 & 0 & 0 & . \\ y_o & 1 & 0 & 0 & . \\ 0 & y_1 & 1 & 0 & . \\ 0 & 0 & y_2 & 1 & . \end{matrix} \quad = Y, \qquad \begin{matrix} 1 & x_o & 0 & 0 & \dots \\ 0 & 1 & x_1 & 0 & \dots \\ 0 & 0 & 1 & x_2 & \dots \\ 0 & 0 & 0 & 1 & \dots \end{matrix} \quad = X.$$

Then a formal multiplication of these matrices gives

$$(11) \quad YX = \begin{matrix} 1 & x_o & 0 & 0 & 0 & \dots \\ y_o & 1+y_o x_o & x_1 & 0 & 0 & \dots \\ 0 & y_1 & 1+y_1 x_1 & x_2 & 0 & \dots \\ 0 & 0 & y_2 & 1+y_2 x_2 & x_3 & \dots \\ 0 & 0 & 0 & y_3 & 1+y_3 x_3 & \dots \end{matrix}$$

Multiplying YX from the left by a diagonal matrix E with elements $e_o, e_1, e_2, \dots$ in the main diagonal, and postulating that EYX shall be equal to M, we find the relations

$$(12) \quad \begin{cases} e_o = 1 \quad , \quad e_o x_o = p_o \\ e_1 y_o = q_o \ , \quad e_1(1+x_o y_o) = 1, \quad e_1 x_1 = p_1 \\ ----------- \quad ----------- \\ e_{n+1}\, y_n = q_n \ , \ e_{n+1}(1 + y_n x_n) = 1, \ e_{n+1}\, x_{n+1} = p_{n+1} \ . \end{cases}$$

We derive from (12) :

$$(13) \qquad y_n = q_n/e_{n+1} \quad , \qquad x_{n+1} = p_{n+1}/e_{n+1} \ ,$$

$$(14) \qquad e_{n+1} = 1 - w_n/e_n = 1 - x_n\, q_n \ .$$

Since $e_o = 1$ and $e_1 = 1 - w_o$ we find for e_{n+1} the continued fraction

$$(15)\quad e_{n+1} = 1 - \cfrac{w_n}{1 - \cfrac{w_{n-1}}{1 - \cfrac{w_{n-2}}{\ddots - \cfrac{w_1}{1 - w_o}}}} .$$

From (15) we see that y_n and x_{n+1} are determined completely by (13), provided that none of the e_n equals zero. We can now find a formal inverse of M by observing that

$$(16)\quad X^{-1} = \begin{matrix} 1 & -x_o, & x_o x_1, & -x_o x_1 x_2, & x_o x_1 x_2 x_3, & \cdots & \cdots \\ 0 & 1 & -x_1, & +x_1 x_2, & -x_1 x_2 x_3, & \cdots \\ 0 & 0 & 1 & -x_2 & x_2 x_3 & \cdots \\ 0 & 0 & 0 & 1 & -x_3 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \end{matrix}$$

We summarize the result by stating

<u>Lemma 2</u>. Let e_n be defined by (15) or by $e_n = z_n/z_{n-1}$ where z_n is defined by (9). Assume that $\lim_{n\to\infty} z_n$ exists and is different from zero. Let

$$(17)\quad C_n = 1 + \frac{w_n}{e_n e_{n+1}} + \frac{w_n w_{n+1}}{e_n e_{n+1}^2 e_{n+2}} + \frac{w_n w_{n+1} w_{n+2}}{e_n e_{n+1}^2 e_{n+2}^2 e_{n+3}} + \cdots ,$$

or, recursively, let

$$(18)\qquad \delta_{n+1} = \frac{(C_n - 1)\, z_n z_{n+1}}{w_n\, z_{n-1}\, z_n} .$$

For $n,m = 0,1,2, \ldots$ let

$$(19)\qquad \begin{aligned} A_m &= (p_o p_1 \ldots p_m)/z_m , \quad A_{-1} = 1 \\ B_m &= (q_o q_1 \ldots q_m)/z_{m+1}, \quad B_{-1} = 0 . \end{aligned}$$

Finally, let us assume that all of the e_n are different from zero. Now, consider the matrix M given by (2), i.e.,

$$(20) \qquad M = (\mu_{n,m}) \qquad (n,m = 0,1,2, \ldots)$$

The inverse of this matrix can be written in the form

$$(21) \qquad M^{-1} = (\mu^*_{n,m}) ,$$

where

$$(22) \qquad \mu^*_{n,m} = C_n/e_n$$

and, for $\ell = 1,2, \ldots$,

$$(23) \qquad \begin{aligned} \mu^*_{n,n+\ell} &= (-1)^{\ell} \; C_{n+\ell} \; A_{n+\ell-1} / (A_{n-1} \, e_{n+\ell}) \\ \mu^*_{n+\ell,n} &= (-1)^{\ell} \; C_{n+\ell} \; B_{n+\ell-1} / (B_{n-1} \, e_n) . \end{aligned}$$

If one of the e_n (or z_n) vanishes, these formulas become meaningless and have to be replaced by more complicated ones. Since it is easy to show that the sum of the squares of the moduli of the elements in $X^{-1} - \underline{I}$ and $Y^{-1} - \underline{I}$ converge ($\underline{I}$ denotes the identity), we can conclude that M^{-1} is a bounded matrix if $\lim_{n \to \infty} z_n$ exists and is different from zero and if $\sum |w_n|$ is finite.

We also mention the following fact, which is proved easily:

Lemma 3. If in the matrix M one element in the main diagonal, say μ_{kk} , is replaced by zero, M^{-1} still exists and can be expressed in the manner of Lemma 2 if we determine e_k, x_k, y_{k-1} by the relations

$$(24) \qquad \begin{aligned} y_{k-1} &= -1/x_{k-1} = -e_{k-1}/p_{k-1} \\ e_k &= q_{k-1}/y_{k-1} = -w_{k-1}/e_{k-1} \\ x_k &= p_k/e_k = -p_k \, e_{k-1}/w_{k-1}. \end{aligned}$$

Here we assume that $w_{k-1} = p_{k-1} \, q_{k-1} \neq 0$, and that the assumptions stated in lemma 2 still hold. For all values of $n \neq k$, the y_{n-1}, e_n, x_n have to be determined as before.

We can use the results of this section to find a periodic solution of the inhomogeneous Mathieu equation

$$y'' + 4(\omega^2 + 2t\cos 2x)y = f(x) \,, \tag{25}$$

where $f(x)$ is, for instance, an even function having period π and having a continuous first derivative. We shall expect (25) to have a uniquely determined even solution $z(x)$ whose period is π, if the homogeneous equation

$$y'' + 4(\omega^2 + 2t\cos 2x)\, y = 0 \tag{26}$$

does not have such a solution. We can find $2(x)$ by putting

$$z(x) = \sum_{n=0}^{\infty} \xi_n \cos 2nx \tag{27}$$

$$f(x) = \sum_{n=0}^{\infty} b_n \cos 2nx \,. \tag{28}$$

Then we find the following equations for the ξ_n:

$$\begin{array}{lcl} -\omega^2\xi_0 - t\,\xi_1 & = & -b_0 \\ -2t\,\xi_0 + (1-\omega^2)\xi_1 - t\,\xi_2 & = & -b_1/1^2 \\ -t\xi_1/2^2 + (1 - \frac{\omega^2}{2^2})\,\xi_2 - t\xi_3/2^2 & = & -b_2/2^2 \\ \text{-- -- -- -- -- -- -- --} & & \end{array} \tag{29}$$

The determinant of the linear equations is exactly

$$-\frac{\omega}{\pi}(\sin\pi\omega)\, C_+ \,,$$

where C_+ is defined by (II.20). The vanishing of $\omega \sin\pi\omega\, C_+$ is a necessary and sufficient condition for the existence of an even solution of period π of equation (2.b). If no such solution exists, we can apply the results of this section to the system (29), and if none of the quantities e_n vanishes, it is not difficult to verify that the solution of (29) for the ξ_n is such that $z(x)$ has a continuous second derivative.

II. Infinite Determinants Related to Periodic Solutions of Mathieu's Equation

We shall write Mathieu's equation in the form

$$y'' + 4(\omega^2 + 2t\cos 2x)\,y = 0 \tag{1}$$

and we shall state the conditions under which, for a given α, $y\exp(-2i\alpha x)$ is a periodic function having period π for at least one solution y of (1). The customary form of (1) is

$$y'' + (a - 2q\cos 2x)\,y = 0, \tag{2}$$

and 2α is usually denoted by μ.

If $\alpha = m/n$, where m,n are co-prime integers, and if $y\exp(-2i\alpha x)$ is periodic and has the period π, then y itself is periodic and has the period $n\pi$. It is known (and will be proved again here) that for $n = 1,2$, there do not exist two linearly independent solutions of (1) both of which have the period $n\pi$. If $n > 2$, however, there will be two such solutions for appropriately chosen values of ω and t.

It is well-known that for given values of ω, t the value of α can be computed from either (4) or (15).

Let y_1, y_2 denote the solutions of (1) which satisfy the initial conditions

$$\begin{aligned} y_1(o) &= 1, & y_1'(o) &= 0 \\ y_2(o) &= 0, & y_2'(o) &= 1. \end{aligned} \tag{3}$$

Then

$$\cos 2\pi\alpha = y_1(\pi;\omega, t), \tag{4}$$

where we mention the parameters ω, t explicitly in the function $y_1(x;\omega,t)$. Since $\exp 2\pi i\alpha$, rather than α itself, describes the behavior of y, we may restrict α in such a way that

$$0 \leq \operatorname{Re}\alpha < 1. \tag{5}$$

Then (4) gives two possible values for α if the right-hand side of the equation is known, since if α is a solution of (4) satisfying (5), then $1 - \alpha$ is also such a solution. However, if $\alpha = \frac{1}{2}$ or $\alpha = 0$ there will be only one value of α satisfying (4) and (5). It is known that for every α satisfying (4),(5), there actually exists a solution y of (1) such that $y(x+\pi) = \exp(2\pi\alpha i)\, y(x)$. We shall prove now:

Lemma 4. Let α be a solution of (4), (5) and let α be different from zero and from 1/2. Let y,z be solutions of (1) such that

$$y(x+\pi) = e^{2\pi i\alpha}\, y(x), \quad z(x+\pi) = e^{2\pi i(1-\alpha)}\, z(x) .$$

Then $y(x)$ and $z(x)$ are linearly dependent. Proof: If y and z are linearly dependent we may assume that $y = \rho z$, where $\rho \neq 0$ is a constant. This gives

$$y(x+\pi) = e^{2\pi i\alpha}\, y(x) = \rho z(x+\pi) = \rho\, e^{-2\pi i\alpha}\, z(x).$$

Therefore

$$y(x) = \rho z(x) = \rho\, e^{-4\pi i\alpha}\, z(x) ,$$

This implies that $\exp(4\pi i\alpha) = 1$ and therefore that either $\alpha = 0$ or $\alpha = \frac{1}{2}$, which contradicts our assumption.

As an immediate consequence of lemma 4 we have:

Theorem 1. If Mathieu's equation (1) has a periodic solution with a period $n\pi$, $n > 2$, all of its solutions have this period.

The possibility that all solutions of (1) may be periodic has been mentioned by Hille [2] . For even values of n, the existence of two solutions of period n for appropriate values of ω and t was proved by Poole [9] . Since it is known* that (4) can be satisfied for any real value of α and infinitely many pairs of values of ω and t, Theorem I provides examples for "transparent media" , i.e., for the Ramsauer effect, in which a plane wave passes through a finite layer of a refracting medium without reflection.

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* $y_1(\pi,\omega,t) = \cos 2\pi\alpha$ is a continuous function of ω and t. This follows directly from Picard's method of solving a differential equations by iteration. The general theory of Mathieu's equation shows (see e.g. McLachlan [5]that $y_1(\pi,\omega,t)$ is real for real values of t and ω and that for a fixed t it takes on the values +1 and -1 infinitely many times for real values of ω. Therefore, $\cos 2\pi\alpha$ takes on energy values between +1 and -1 infinitely many times.

For a more detailed account of this matter see Section VII.

The following statements about solutions of (1) which are periodic and have the period π or 2π are standard results in the theory of Mathieu functions.

I. A solution which has the period π is either even or odd and can be represented in one of the following two ways:

$$(6) \qquad ce_+(x) = \sum_{r=0}^{\infty} A_{2r} \cos 2rx \,, \qquad se_+(x) = \sum_{r=1}^{\infty} B_{2r} \sin 2rx \,.$$

II. A solution which has the period 2π but not the period π is either even or odd and can be represented in one of the following two ways:

$$(7) \qquad ce_-(x) = \sum_{r=0}^{\infty} A_{2r+1} \cos (2r+1)x \,, \quad se_-(x) = \sum_{r=0}^{\infty} B_{2r+1} \sin (2r+1)\, x \,.$$

III. If for a certain pair of values of ω and t there exists a periodic solution of (1) which is of type (6) or (7), the multiples of this solution are the only periodic solutions of (1).

We shall now prove I, II and III by using infinite determinants.

Assuming that (1) has a solution of the type

$$(8) \qquad y = e^{2i\alpha x} \sum_{m=-\infty}^{+\infty} c_m \, e^{2imx} \,,$$

to which two term-by-term differentiations can be applied, the c_m have to satisfy the equations

$$(9) \quad 4\, t\, c_{m-1} + 4[\omega^2-(m+\alpha)^2]\, c_m + 4 + 4t\, c_{m+1} = 0 \,.$$

$$(m = 0, \pm 1, \pm 2, \dots)$$

Dividing (9) by $4(\omega^2-m^2)$, the determinant of the resulting system of linear equations

$$(10) \quad \frac{t}{\omega^2-m^2}\, c_{m-1} + \frac{\omega^2-(m-\alpha)^2}{\omega^2-m^2}\, c_m + \frac{t}{\omega^2-m^2}\, c_{m+1} = 0$$

exists and shall be denoted by

$$(11) \qquad \Delta(\alpha\,;\,\omega, t) \,.$$

We write

$$(12) \qquad \Delta_o(\omega,t) = \Delta(0;\omega,t) ,$$

and then there exists the relation (see for instance Whittaker and Watson [14])

$$(13) \qquad \Delta(a;\omega,t) = \Delta_o(\omega,t) - \left(\frac{\sin\pi a}{\sin\pi\omega}\right)^2 .$$

Here $\Delta_o(\omega,t)$ can be written as a determinant ${}_{\infty}^{-\infty}M(p_\nu,q_\nu)$ of type (I.1);

$$(14) \qquad p_\nu = q_{\nu-1} = \frac{t}{m^2-\omega^2} .$$

A necessary and sufficient condition for the existence of a solution of (1) of type (6) which can be differentiated twice term by term is the vanishing of $\sin^2\pi\omega\,\Delta(a,\omega,t)$. Therefore we obtain from (4) and (13):

$$(15) \qquad \cos 2\pi a = y_1(\pi,\omega,t) = 1 - 2\,\Delta_o(\omega,t)\sin^2\pi\omega .$$

If (1) has a periodic solution $ce_+(x)$ of the type (6), the A_{2r} must satisfy the set of linear equations

$$(16) \qquad \begin{aligned} &4\omega^2 A_o + 4t\,A_2 = 0 \\ &8t\,A_o + 4(\omega^2-1)\,A_2 + 4t\,A_4 = 0 \\ &4t\,A_{2r-2} + 4(\omega^2-r^2)\,A_{2r} + 4t\,A_{2r+2} = 0 , \\ &\qquad (r = 2,3,4, \ldots) . \end{aligned}$$

Similarly, the existence of periodic solutions of the type se_+, ce_-, se_- leads to the following systems of linear equations:

$$(17) \qquad \begin{aligned} &4(\omega^2-1)\,B_2 + 4t\,B_4 = 0 , \\ &4t\,B_{2r-2} + 4(\omega^2-r^2)\,B_{2r} + 4t\,B_{2r+2} = 0 , \\ &\qquad (r = 2,3,4, \ldots) ; \end{aligned}$$

$$(18)\qquad \begin{aligned} &4\left\{\left[\omega^2-\left(\tfrac{1}{2}\right)^2\right]+t\right\}A_1+4t\,A_3=0\,,\\ &4t\,A_{2r-1}+4\left[\omega^2-\left(r+\tfrac{1}{2}\right)^2\right]A_{2r+1}+4t\,A_{2r+3}=0\,,\\ &\qquad\qquad (r=1,2,3,\ldots); \end{aligned}$$

$$(19)\qquad \begin{aligned} &4\left\{\left[\omega^2-\left(\tfrac{1}{2}\right)^2\right]-t\right\}B_1+4t\,B_3=0\,,\\ &4t\,B_{2r-1}+4\left[\omega^2-\left(r+\tfrac{1}{2}\right)^2\right]B_{2r+1}+4t\,B_{2r+3}=0\,,\\ &\qquad\qquad (r=1,2,3,\ldots)\,. \end{aligned}$$

If we divide the equations (16), (17), (18), (19) by certain factors in such a manner that the matrix of the resulting system of linear equations becomes the unit matrix for $t = 0$, we arrive at the following infinite determinants

$$(20)\qquad C_+=\begin{vmatrix} 1 & t/\omega^2, & 0, & 0, & \ldots \\ \dfrac{2t}{\omega^2-1}, & 1, & \dfrac{t}{\omega^2-1}, & 0, & \ldots \\ 0 & \dfrac{t}{\omega^2-2^2}, & 1 & \dfrac{t}{\omega^2-2^2}, & \ldots \\ 0 & 0 & \dfrac{t}{\omega^2-3^2} & 1, & \ldots \\ \hline \end{vmatrix}$$

$$(21)\qquad S_+=\begin{vmatrix} 1, & \dfrac{t}{\omega^2-1^2}, & 0, & 0, & \ldots \\ \dfrac{t}{\omega^2-2^2}, & 1, & \dfrac{t}{\omega^2-2^2}, & 0, & \ldots \\ 0 & \dfrac{t}{\omega^2-3^2}, & 1, & \dfrac{t}{\omega^2-3^2}, & \ldots \\ 0 & 0 & \dfrac{t}{\omega^2-4^2}, & 1 & \ldots \\ \hline \end{vmatrix}$$

$$(22)\quad C_- = \begin{vmatrix} 1 + \frac{t}{\omega^2-(\frac{1}{2})^2} , & \frac{t}{\omega^2-(\frac{1}{2})^2} , & 0 , & 0 , & \cdots \\ \frac{t}{\omega^2-(\frac{3}{2})^2} , & 1 , & \frac{t}{\omega^2-(\frac{3}{2})^2} , & 0 , & \cdots \\ 0 , & \frac{t}{\omega^2-(\frac{5}{2})^2} , & 1 , & \frac{t}{\omega^2-(\frac{5}{2})^2} , & \cdots \\ 0 , & 0 , & \frac{t}{\omega^2-(\frac{7}{2})^2} , & 1 , & \cdots \\ - - - & - - - & - - - & - - - & - - - \end{vmatrix}$$

$$(23)\quad S_- = \begin{vmatrix} 1 - \frac{t}{\omega^2-(\frac{1}{2})^2} , & \frac{t}{\omega^2-(\frac{1}{2})^2} , & 0 , & 0 , & \cdots \\ \frac{t}{\omega^2-(\frac{3}{2})^2} , & 1 , & \frac{t}{\omega^2-(\frac{3}{2})^2} , & 0 , & \cdots \\ 0 , & \frac{t}{\omega^2-(\frac{5}{2})^2} , & 1 , & \frac{t}{\omega^2-(\frac{5}{2})^2} , & \cdots \\ 0 , & 0 , & \frac{t}{\omega^2-(\frac{7}{2})^2} , & 1 , & \cdots \\ - - - & - - - & - - - & - - - & - - - \end{vmatrix}$$

Of these determinants, C_+ and S_+ are of the type ${}^{\,0}_{\infty}D$ of Section I, and C_- and S_- differ from this type only in the first element of the main diagonal.

We shall now prove:

Theorem 2.

(24) $$\sin^2 \pi\omega\, \Delta_0(\omega,t) = (\sin \pi\omega\, C_+)\,(\sin \pi\omega S_+)$$

(25) $$\sin^2 \pi\omega\, [\Delta_0(\omega,t)] - 1 = \sin^2 \pi\omega\, [\Delta(\tfrac{1}{2};\omega,t)] = -\cos^2 \pi\omega\, C_-\, S_- .$$

If t is a fixed number different from zero, no more than one of the functions

$(\sin \pi\omega)\, C_+$, $(\sin \pi\omega)\, S_+$, $(\cos \pi\omega)\, C_-$ $(\cos \pi\omega)\, S_-$

vanish for any given value of ω. Equation (24) establishes the well-known fact that the condition $\sin^2 \pi\omega\, \Delta_0(\omega,t) = 0$ for the existence of a solution of period π is satisfied if and only if one of the conditions $(\sin\pi\omega)C_+ = 0$ or $(\sin\pi\omega)S_+ = 0$ is satisfied, that is, if a solution ce_+ or se_+ exists. Equation (25) establishes the corresponding fact for the solutions of period 2π. However, (24) and (25) state more than this: they establish a factorization of an entire function (namely $\sin^2 \pi\omega\, \Delta_0(\omega,t)$) into the product of two other entire functions. We can derive (22) from the recurrence formulas (I.6), (I.7). We shall use the notations of Section I and in particular we shall denote subdeterminants by double subscripts. Then, if we put

(26) $$p_{-\nu} = p_\nu = t/(\omega^2-\nu^2)\ ,\ q_\nu = p_{\nu+1}\ ,\ w_\nu = p_\nu q_\nu\ ,$$

remembering that $p_{-\nu} = p_\nu$, $w_{-\nu} = w_{\nu-1}$, we find that the subdeterminant of $\Delta_0(\omega,t)$ which corresponds to ${}_{-1}^{-\infty}D$ in (I.6) would be C_+ if in the first column of C_+ we substitute $t/(\omega^2-1)$ for $2t(\omega^2-1)$. Since (I.7) gives

(27) $$\begin{aligned} C_+ &= S_+ - 2w_0\, S_{+,1,1} \\ S_+ &= S_{+,1,1} - w_1\, S_{+,1,1;2,2} \end{aligned}$$

we find from (I.6) that

(28) $$\begin{aligned} \Delta_0(\omega,t) &= (S_+ - w_0 S_{+,11})\,(S_{+,11} - w_1\, S_{+,11,22}) - w_0 S_+ S_{+,11} \\ &= S_+(S_+ - 2w_0\, S_{+,11}) = S_+\, C_+\ , \end{aligned}$$

which proves (24). To prove (25) we observe first that

(29) $$\left\{\frac{\sin\pi\omega}{\cos\pi\omega}\right\}^2 = - \prod_{n=-\infty}^{+\infty} \frac{\omega^2 - n^2}{\omega^2 - (n+\frac{1}{2})^2}$$

if the product is defined as

$$\lim_{N\to\infty} \prod_{n=-N}^{+N} .$$

Multiplying equation (10) (for $a = \frac{1}{2}$) by $(\omega^2-m^2)/(\omega^2-(m+\frac{1}{2})^2)$, we find from (29) that

(30) $$\Delta^* = tq^2\pi\omega\, \Delta(\tfrac{1}{2};\ \omega,t)$$

is an infinite determinant of type (I.1) if we specify

(31) $$p_\nu = t/\left[\omega^2 - (\nu+\tfrac{1}{2})^2\right], \quad q_\nu = p_{\nu-1}\ ;\ w_\nu = p_\nu q_\nu .$$

We then have

(32) $$p_{-\nu} = p_{\nu-1},\ q_{-\nu} = p_{\nu-2},\ w_{-1} = p_o^2 .$$

Defining the determinant T by

(33) $$T = \begin{vmatrix} 1 & p_o & 0 & 0 & 0 \\ p_1 & 1 & p_1 & 0 & 0 \\ 0 & p_2 & 1 & p_2 & 0 \\ 0 & 0 & p_3 & 1 & p_3 \\ - & - & - & - & - \end{vmatrix},$$

where the p_ν are given by (31), we find that

(34) $$S_- = T - p_o\, T_{1,1}, \qquad C_- = T + p_o\, T_{1,1} .$$

On the other hand, if we substitute Δ^* for ${}_{\infty}^{-\infty}D$ in (I.6') we obtain

$$(35)\qquad {}_{-2}^{-\infty}D = T\,, \qquad {}_{-3}^{-\infty}D = T_{1,1}\,, \quad {}_{\infty}^{1}D = T_{1,1}\,, \quad {}_{\infty}^{2}D = T_{1,1;2,2}$$

and therefore, from (I.6), (I.7), and (32) :

$$(36)\qquad \begin{aligned} \Delta^* &= T\,T_{1,1} - p_o^2\,T_{1,1}^2 - w_o\,T\,T_{1,1;2,2} \\ &= T(T_{1,1} - w_o\,T_{1,1;2,2}) - p_o^2\,T_{1,1}^2 \\ &= T^2 - p_o^2\,T_{1,1}^2 = S_-\,C_-\,, \end{aligned}$$

This proves equation (25) of Theorem 2.

To prove the statement of Theorem 2 about the non-existence of common zeros of the four functions $(\sin\pi\omega)C_+$, etc., we observe first that (24) and (25) exclude the possibility of $(\sin\pi\omega)C_+$ or $(\sin\pi\omega)C_-$ having a zero in common with $(\cos\pi\omega)C_-$ or $(\cos\pi\omega)S_-$. The fact that S_+ and C_+ cannot have a zero in common unless $t = 0$ can be derived from (27) as follows: If $S_+ = C_+ = 0$ and $t \neq 0$, we would also have $w_o \neq 0$, $w_1 \neq 0$ and consequently $S_{+;1,1} = S_{+;1,1;2,2} = 0$. By an application of (I.7) to $S_{+;1,1}$ and $S_{+};1,1;2,2$ this would lead to $S_{+;1,1;2,2;3,3} = 0$ and so on, which is impossible since the subdeterminants of this type approach unity. The possibility that $(\sin\pi\omega)\,C_+$ and $(\sin\pi\omega)\,S_+$ could have a common root at $\omega = m\pi$, $m = 0, \pm 1, \pm 2, . .$ can be excluded for, if there were such a common root, the row of $(\sin\pi\omega)\,C_+$ which is labeled by $|m|$ would become, for $\omega \to \pm m$ $(m \neq 0)$,

$$0,\ 0,\ \frac{\pm t\pi}{2m}\,,\ 0,\ \frac{\pm t\pi}{2m}\ 0,\ 0\ .\ .\ .$$

where the zero in the center is the element of the main diagonal. The corresponding recurrence formula would then read

$$S_{+,1,1;\ \dots\ ;\ |m|-2,\ |m|-2} = \lambda\, S_{+,1,1;\dots;\ |m|,\ |m|},$$

where $\lambda \neq 0$. The result is a contradiction as before.

The non-existence of common zeros of $(\cos\pi\omega)\, C_-$ and $(\cos\pi\omega)\, S_-$ can be proved in the same manner.

III. A Majorant for Mathieu's Determinants

There are two methods of obtaining a majorant for the absolute value of a determinant of type (I.1) or (I.2). The first is to apply (I.5), which gives $\prod(1+|w_\nu|)$ as an upper bound for the determinant. The second method is to apply Hadamard's inequality which states that the modulus of the determinant of the elements $a_{\nu,\mu}$ is at most equal to the product of all s_ν, where

$$(1)\qquad s_\nu = \Big\{\sum_\mu |a_{\nu,\mu}|^2\Big\}^{1/2} \leqq \sum_\mu |a_{\nu,\mu}| .$$

Also, we may observe that for a determinant D of type (I.1) or (I.2),

$$(2)\qquad \Big|\prod_\nu (1+|w_\nu|) - 1\Big| \geqq |D-1|$$

since the terms on the left-hand side in (2) majorize the moduli of the terms of D-1 if D is expanded in a sum of type (I.4). Applying (I.5) to $\Delta_0(\omega,t)$, which is defined by (II.12),(II.14), we find

$$(3)\quad |\Delta_0(\omega,t)\sin^2\pi\omega| \leqq |\sin^2\pi\omega| \prod_{n=-\infty}^{\infty} 1 + \left|\frac{t^2}{(n^2-\omega^2)\,|(n+1)^2-\omega^2|}\right|$$

$$\frac{\pi^4}{|\sin\pi\omega|^2} \prod_{n=-\infty}^{\infty} \left\{ \frac{|(n^2-\omega^2)((n+1)^2-\omega^2)| + |t|^2}{n^{*2}\ (n+1)^{*2}} \right\},$$

where $n^* = n$ for $n \neq 0$, $0^* = 1$, and $(n+1)^* = n+1$ for $n \neq -1$, $0^* = 1$. Now

$$(4) \qquad |(n^2-\omega^2)((n+1)^2-\omega^2)| + |t|^2 \leq (n^2 + |\omega^2| + |t^2/\omega^2|)((n+1)^2 + |\omega^2|),$$

and therefore we find from (3) :

$$(5) \quad |\Delta_o(\omega,t)\sin^2\pi\omega| \leq \frac{\left\{\sinh\pi\left[|\omega^2| + |t^2/\omega^2|\right]^{1/2}\sinh\pi|\omega|\right\}^2}{|\sin\pi\omega|^2}.$$

Combining (5) and (2) we find that, for a fixed value of t and for $\omega = i\lambda$, λ real,

$$(6) \qquad \lim_{\lambda \to \infty} |\Delta_o(i\lambda,t)| = 1.$$

To apply Hadamard's inequality we observe first that the expansion of $\sin^2\pi\omega$ into an infinite product shows that the n^{th} row $(n = 0,\pm 1, \pm 2, \ldots)$ of $\Delta_o(\omega,t)\sin^2\pi\omega$ can be written as follows:

For $n \neq 0$,

$\ldots, 0, 0, 0, -t/n^2, 1 - \frac{\omega^2}{n^2}, -t/n^2, 0, 0, 0, \ldots$, and for $n = 0$,

$\ldots, 0, 0, 0, \pi^2 t, \pi^2\omega^2, \pi^2 t, 0, 0, 0, \ldots$

Therefore we find from (2):

$$(7) \quad |\Delta_o(\omega,t)\sin^2\pi\omega| \leq \pi^2(|\omega|^2 + 2|t|)\left\{\prod_{n=1}^{\infty}\left(1 + \frac{|\omega^2| + 2|t|}{n^2}\right)^2\right\}$$

$$= \sinh^2\pi\left[|\omega|^2 + 2|t|\right]^{\frac{1}{2}}.$$

Applying the same method to C_+, S_+, C_-, S_- of Section II, we obtain results which may be summarized as follows:

Lemma 5. For all values of ω and t we have the inequalities

(8) $$|\sin \pi\omega|\,|C_+| \leqq \frac{[|\omega^2| + 2|t|]^{\frac{1}{2}}}{|\omega|} \sinh\left[\pi\,(|\omega|^2 + 2|t|)^{\frac{1}{2}}\right],$$

(9) $$\left|\frac{\sin \pi\omega}{\omega} S_+\right| \leqq \frac{\sinh\left[\pi\,(|\omega^2| + 2|t|)^{\frac{1}{2}}\right]}{[|\omega^2| + 2|t|]^{\frac{1}{2}}},$$

(10) $$|\cos \pi\omega|\,|C_-| \leqq \cosh\left[\pi(|\omega|^2 + 2|t|)^{\frac{1}{2}}\right],$$

(11) $$|\cos \pi\omega|\,|S_-| \leqq \cosh\left[\pi(|\omega|^2 + 2|t|)^{\frac{1}{2}}\right].$$

The inequality (7) is an immediate consequence of (8) and (9). Lemma 5, together with the relation (6), implies a result found by Schaefke (theorem VII, p. 39 in [11]) about the order of growth of $(\sin^2 \pi\omega)\,\Delta_0(\omega t)$.

IV. Expressions in Terms of Infinite Determinants for the Values of the Solutions of Mathieu's Equation for $x = \frac{\pi}{2}$ and $x = \pi$.

In this section the following result will be proved:

Theorem 3. Let $y_1(x;\omega,t)$ and $y_2(x;\omega,t)$ denote the even and odd solutions of Mathieu's equations (II.1) and assume with the initial values (1,0) and (0,1) for y, y' at $x = 0$. Let C_+, S_+, C_-, S_- be the infinite determinants defined by (II.20) to (II.23). Then

(1) $$y_1(\tfrac{\pi}{2};\,\omega,t) = (\cos \pi\omega)\, C_-$$

(2) $$y_1'(\tfrac{\pi}{2};\,\omega,t) = -2\omega(\sin\pi\omega)\, C_+$$

(3) $$y_2(\tfrac{\pi}{2};\omega,t) = \frac{\sin \pi\omega}{2\omega}\, S_+$$

(4) $$y_2'(\tfrac{\pi}{2};\omega,t) = (\cos\pi\omega)\, S_-$$

These formulas may be supplemented by the elementary relations

(5) $2y_1'(\frac{\pi}{2};\omega,t)\, y_2(\frac{\pi}{2};\omega,t) = y_1(\pi;\omega,t) - 1$

(6) $2y_1(\frac{\pi}{2};\omega,t)\, y_2'(\frac{\pi}{2};\omega,t) = y_1(\pi;\omega,t) + 1$

(7) $y_1(\pi;\omega,t) = y_2'(\pi,\omega,t)$,

of which (5) and (6) have been found by Schaefke [10] and (7) is stated in McLachlan's book [5].

To prove (1) to (4) we must prove the following facts:

1. Necessary and sufficient conditions for Mathieu's equation (II.1) to have solutions of the following types:

even, with period 2π
even, with period π
odd, with period π
odd, with period 2π

are

(8)
$$y_1(\tfrac{\pi}{2};\omega,t) = 0$$
$$y_1'(\tfrac{\pi}{2};\omega,t) = 0$$
$$y_2(\tfrac{\pi}{2};\omega,t) = 0$$
$$y_2'(\tfrac{\pi}{2};\omega,t) = 0$$

respectively.

Proof: Assume for instance that $y(x) = y(x+2\pi)$ and $y(-x) = y(x)$.

Then we derive from $y = c_1y_1(x) + c_2y_2(x)$ (where c_1, c_2 are constant values) that $c_1 = y(0)$, $c_2 = 0$.

Therefore $y = y_1(x)$ is the even solution with period 2π. Because of the periodicity of the coefficients of Mathieu's equation, $y(x+\pi)$ is also a solution if $y(x)$ is one.

Therefore we find that

$$y_1(x+\pi) = y_1(\pi)\, y_1(x) + y_1'(\pi)\, y_2(x)$$
$$y_2(x+\pi) = y_2(\pi)\, y_1(x) + y_2'(\pi)\, y_2(x) .$$

The matrix
$$M = \begin{pmatrix} y_1(\pi) & y_1'(\pi) \\ y_2(\pi) & y_2'(\pi) \end{pmatrix}$$
has a determinant which is the Wronskian of the fundamental solutions $y_1(x)$ and $y_2(x)$ and is equal to unity. M^2 must be a matrix in which the elements of the first row are 1 and 0 since $y_1(x+2\pi) = y_1(x)$. From these conditions we can derive the fact that
$$y_1(\pi)\ |y_1(\pi) + y_2'(\pi)\ | = 2,\ y_1'(\pi) = 0,\ \left| y_1(\pi) \right|^2 = 1 .$$

Therefore we have $y_1(\pi) = -1$, $y_2'(\pi) = -1$. (We discard the other possibility, $y_1(\pi) = y_2'(\pi) = 1$, because it would mean that $y_1(x+\pi) = y_1(x)$ and hence that y_1 would be already of period π.) Therefore we find
$$y_1(x+\pi) = -y_1(x) .$$
For $x = -\pi/2$, we find that $y_1(-\pi/2) = y_1(\pi/2) = -y_1(\pi/2) = 0$. On the other hand if $y_1(\pi/2) = 0$, we also have $y_1(-\pi/2) = 0$, and since there must exist a relation
$$y_1(x+\pi) = \gamma_1 y_1(x) + \gamma_2 y_2(x) ,$$
we find, putting $x = 0$ and $x = -\frac{\pi}{2}$, that
$$\gamma_1 = y_1(\pi) ,\ \gamma_2 = 0 .$$
Therefore
$$y_1(0) = y_1(-\pi+\pi) = y_1(\pi)\ y_1(-\pi) = \left\{ y_1(\pi) \right\}^2 = 1 ,$$
and therefore $y_1(x+\pi) = -y(x)$ unless $\gamma_1 = 1$, in which case $y_1(x)$ has the period π. But then
$$y_1'(x+\pi) = y_1'(x)$$
gives for $x = -\frac{\pi}{2}$
$$y_1'(\pi/2) = y_1'(-\tfrac{\pi}{2}) .$$

Since $y_1'(x)$ is an odd function of x_1 we would have $y_1'(\pi/2) = 0$, which is incompatible with $y_1(\pi/2) = 0$. Therefore, $y_1(x+\pi) = -y_1(x)$ follows from $y_1(\pi/2) = 0$, and this completes the proof of our statement 1.

The next statement which we shall prove is

2. Let $\lambda = 4\omega^2$ and consider the equations (8), for a fixed real value of t, as equations for λ. Then all the roots of these equations are simple ones.

Proof: We use the fact that for $y_1(\pi/2) = 0$,

$$(9) \qquad \frac{\partial}{\partial\lambda} y_1\left(\frac{\pi}{2}\right) = -\frac{1}{y_1'(\pi/2)} \int_0^{\pi/2} \left\{ y_1(\xi) \right\}^2 d\xi \,.$$

Equation (9) and similar formulas for the other functions in (8) can easily be derived from Mathieu's equation. Now if t is real, $\left(y_1(\xi)\right)^2$ is non-negative, and this proves statement 2.

Finally, we prove the following statement about the left-hand sides in (8):

3. Let x_o be any fixed (real or complex) value of x and let $y(x;\omega,t)$ be any solution of Mathieu's equation (II.1). Then there exist numbers A and B independent of ω such that

$$(10) \qquad |y(x_o;\omega,t)| \leqq A\, e^{B|\omega|} \,.$$

The proof is part of the well-known method of solving (II.1) by iteration. Choosing for $\eta_o(x)$ a solution of $y'' + 4\omega^2 y = 0$ which satisfies the correct initial conditions and defining $y_n(x)$ for $n = 1,2,3, \ldots$ as that solution of

$$\eta_n'' + 4\omega^2\eta_n = -8t\cos 2x\, \eta_{n-1}$$

which satisfies the correct initial conditions, we can easily see that

$y = \lim_{n\to\infty} \eta_n$ satisfies (10).

The corresponding facts concerning $(\cos\pi\omega)\,C_-$ etc are:

1a. A necessary and sufficient condition for Mathieu's equation to have an even solution of period 2π (but not of period π) is

$$(\cos\pi\omega)\,C_- = 0 \ . \tag{10a}$$

2a. For any fixed real value of t, the roots of (10a) are simple ones, if ω^2 is introduced as a new variable.

3a. $(\cos\pi\omega)\,C_-$ satisfies (III.10).

Statement 1a. has been proved elsewhere; see for instance Whittaker and Watson [14]. Statement 3a. was proved in Section III. Statement 2a. can be proved as follows:

Using (II.15), (II.25) and (6) we find:

$$y_1(\pi;\omega,t) + 1 = 2 - 2\,\Delta_o(\omega,t)\,\sin^2\pi\omega \tag{11}$$

$$= 2\cos^2\pi\omega\,S_-C_- = 2y_1(\tfrac{\pi}{2};\omega,t)\,y_2'(\tfrac{\pi}{2};\omega,t) \ .$$

Now $y_1(\pi/2)$ vanishes if and only if $(\cos\pi\omega)\,C_-$ vanishes, and $y_2'(\pi/2)$ vanishes if and only if $(\cos\pi\omega)\,S_-$ vanishes, since the vanishing of any of these quantities is a necessary and sufficient condition for the existence of a solution of Mathieu's equation with one of the properties described before eq. (8). Now $y_1(\pi/2;\omega,t)$ and $y_2'(\pi/2;\omega,t)$ have simple zeros (with respect to ω^2) for any given real value of t. Therefore the quotients

$$\frac{(\cos\pi\omega)\,C_-}{y_1(\pi/2;\omega,t)} \ , \qquad \frac{(\cos\pi\omega)\,S_-}{y_2'(\pi/2;\omega,t)} \tag{12}$$

are entire functions of ω^2. Since their product is unity, neither of them can have any zeros. Therefore, the multiplicity of the zeros of the numerators cannot be higher than the multiplicity of the zeros of the denominators. This proves statement 2a.

To prove equations (1) to (4) of theorem 3, we can apply a theorem in the theory of entire (or meromorphic) functions the proof of which is found for instance in Nevanlinna [7] (p.205-213) or Titchmarsh [13]. This theorem shows that from statements 3 and 3a the following representations for $y_1(\pi/2;\omega,t)$ and $(\cos \pi\omega)\, C_-$ can be derived:

(13) $$y_1(\pi/2;\omega,t) = \omega^{2\alpha} a(t) \prod_{\nu=1} \left(1 - \frac{\omega^2}{\alpha_\nu}\right)$$

(14) $$(\cos \pi\omega)\, C_- = \omega^{2\alpha} a'(t) \prod \left(1 - \frac{\omega^2}{\alpha_\nu}\right),$$

where α is an integer, the α_ν are the zeros of the left-hand sides in (13) and (14), and where ω^2 is considered as the independent variable. The functions $a(t)$ and $a'(t)$ are independent of ω and will be discussed presently. For any given real t, α is either one or zero , depending on whether $\omega = 0$ is or is not a zero of the left-hand sides in (13), (14).

To show that $a(t) = a'(t)$, we may let $\omega \to i\infty$. In this case we have from Section III:

(15) $$\lim_{\omega \to i\infty} C_- = 1,$$

and from the argument used in deriving statement 3 we can show that

(16) $$\lim_{\omega \to i\infty} e^{i\pi\omega}\, y_1(\pi/2,\omega,t) = 1/2 .$$

Therefore, the quotients in (12) tend to unity if $\omega \to i\infty$. This completes the proof of equation (1) for real values of t, and for complex values of t we obtain the same result by analytic continuation with respect to t. The relation (2) to (4) of theorem 3 can be proved in the same manner.

We can use Theorem 1 to compute the first terms of the expansions of C_+, C_-, S_+, S_- in a series of powers of t. If we define $\eta(x)$ by

(17) $$\eta(x) = y_1(x) + 2i\omega\, y_2(x),$$

we find that

(18) $$\eta(x) = \eta_0(x) + t\,\eta_1(x) + t^2 \eta_2(x) + \ldots ,$$

where

(19) $$\eta_o(x) = e^{2i\omega x}$$

and

(20) $$\eta_n(x) = -\frac{4t}{\omega}\int_o^x \sin\ 2\omega(x-\xi)\ \cos 2\xi\ \eta_{n-1}(\xi)\, d\xi\ .$$

The explicit formulas become rather complicated for $n>2$ and for arbitrary values of x, but they become simpler if x is a multiple of $\pi/2$. The results derived from (1) to (7) and (17) to (19) are:

(21) $$C_- = 1 + \frac{4t}{4\omega^2-1} + \frac{t^2}{\omega(4\omega^2-1)}\left(\frac{8\omega}{4\omega^2-1} + \pi\tan\pi\omega\right) + \cdots$$

(22) $$S_+ = 1 + \frac{t^2}{\omega^2}\left(\frac{1}{\omega^2-1} - \frac{\pi\omega}{4\omega^2-1}\ \operatorname{ctg}\pi\omega\right) + \cdots$$

(23) $$C_+ = 1 - \frac{t^2}{\omega}\left(\frac{1}{\omega(\omega^2-1)} + \frac{\pi}{4\omega^2-1}\ \operatorname{ctg}\ \pi\omega\right) + \cdots$$

(24) $$S_- = 1 - \frac{4t}{4\omega^2-1} + \frac{t^2}{4\omega^2-1}\left(\frac{8}{4\omega^2-1} + \frac{\pi}{\omega}\tan\ \pi\omega\right) + \cdots$$

The series converge for all values of t. It can be shown that the coefficient of t^n in the expansion of any one of the expressions (1) to (4) is of the type

$$R_1(\omega)\cos\omega\pi + R_2(\omega)\sin\omega\pi\ ,$$

where R_1 and R_2 are rational functions of ω which will have no poles except for $\omega = 0, \pm 1, \ldots, \pm(n-1)$ and $\omega = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm(n-\frac{1}{2})$.

Also, for $|\omega| \to \infty$,

$$\lim_{\omega\to\infty} \omega^k R(\omega)$$

exists, where R denotes either R_1 or R_2 and where $k = n+1$ for the expression in (3), $k = n$ for the expressions in (1) and (4), and $k = n-1$ for the expression in (2).

V. Fourier Transforms with Respect to the Parameter ω.

In this section the notations of the previous sections will be used. We may summarize the results as follows:

Theorem 4. *Let* $y(x;\omega,t)$ *be a solution of Mathieu's equation with the initial values*

(1) $$y(0;\omega,t) = a, \qquad y'(0;\omega,t) = b.$$

Then we have for real values of x and ω:

(2) $$y(x;\omega,t) = a \cos 2\omega x + \int_{-x}^{+x} e^{2i\omega\phi} H(x;\phi,t)\, d\phi ,$$

where for all t, $H(x;\phi,t)$ *can be expanded in a series of powers of t. The first terms of the expansion are*

(3) $$H(x;\phi,t) = \frac{1}{2} + t\left[-2a \sin x \cos\phi + 2b \cos x\, (\cos x - \cos\phi\right] + t^2[\,.\;.\;.\,] .$$

In particular, if $a = 0$, $b = 1$, *we find that*

(4) $$y_2(x;\omega,t) = \int_{-x}^{x} e^{2i\omega\phi} G(x,\phi,t)\, d\phi ,$$

where G can be expanded in a power series in t which is everywhere convergent. As a function of the real variables ϕ and x, G satisfies the partial differential equation

(5) $$\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial \phi^2} + 8t \cos 2x\, G = 0.$$

The initial conditions for G and $G_\phi = \frac{\partial G}{\partial \phi}$ are:

(6) $$G(x;\pm x,t) = \frac{1}{2} , \qquad G_\phi(x;\pm x,t) = \pm\, t \sin x \cos x .$$

The proof of Theorem 4 can be derived from Theorem X , p.13 in the book by Paley and Wiener [8] . According to this theorem, the following two classes of functions are identical:

(I) The class of all entire functions F(z) satisfying

(7) $$|F(z)| = o(e^{A|z|}) ;$$

(II) the class of all entire functions of the form

$$F(z) = \int_{-A}^{+A} f(u)\, e^{iuz} du, \tag{8}$$

where $f(u)$ belongs to L_2 over $(-A,A)$.

It may suffice to prove (4), (5) and (6). Putting

$$u_o(x,\omega) = \frac{\sin 2\omega x}{2\omega} \tag{9}$$

and

$$u_n(x,\omega) = -\frac{4}{\omega}\int_0^x \sin\ 2\omega(x-\xi)\ \cos 2\xi\, u_{n-1}(\xi)\, d\xi\ , \tag{10}$$

we find

$$y_2(x,\omega,t) = \sum_{n=0}^{\infty} u_n t^n\ , \tag{11}$$

where the series on the right-hand side converges for all values of t. We derive from (10) that for $\omega \to \pm\infty$,

$$|u_n(x,\omega)| = O(\omega^{-n-1})\ . \tag{12}$$

Therefore

$$g_n(x,\phi) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\omega\phi}\ u_n(x,\omega)\, d\omega \tag{13}$$

exists and is (n-1) times differentiable with respect to ϕ with a continuous (n-1)-st derivative. Also, we find from (9) and (10) by induction that, as a function of the complex variable ω,

$$|u_n(x,\omega)| = o(e^{2|x\omega|})\ . \tag{14}$$

Therefore, Paley and Wiener's theorem proves that $g_n(x,\phi)$ vanishes for real x, ϕ if $|x| > |\phi|$, and then we find (13) by Fourier's inversion formula

$$u_n(x,\omega) = \int_{-x}^{x} e^{2i\omega\phi}\ g_n(x,\phi)\, d\phi\ . \tag{15}$$

From the differentiability of $g_n(x,\phi)$ with respect to ϕ it also follows that for $n \geqq 2$

(16) $$\frac{\partial g_n}{\partial \phi} \;,\; \frac{\partial^2 g_n}{\partial \phi^2}$$

vanish for $\phi = \pm x$. By substituting the expressions in (15) for u_n and u_{n-1} into equation (10) we find after an application of Fourier's inversion formula:

(17) $$g_n(x,\phi) = -\frac{4}{\pi}\int_{-\infty}^{\infty} e^{-2i\omega\phi}\,d\omega \int_0^x \frac{\sin 2\omega(x-\xi)}{\omega}\cos 2\xi\, d\xi \int_{-\xi}^{\xi} e^{2i\omega\psi} g_{n-1}(\xi,\psi)\,d\psi \; .$$

Exchanging the order of integrations and carrying out the integration with respect to ω we obtain the following result: Let

(18) $$L(x,\xi,\phi\;\;) = \int_{-\infty}^{+\infty} e^{2i\omega(\psi-\phi)} \sin 2\omega(x-\xi)\,\frac{d\omega}{\omega}\;;$$

then

(19) $$g_n(x,\phi) = -\frac{4}{\pi}\int_0^x d\xi \int_{\xi}^{\xi} d\psi\; L(x,\xi,\phi,\psi)\; g_{n-1}(\xi,\psi)\cos 2\xi \quad .$$

If the integrals from $-\infty$ to $+\infty$ in (17) and (19) are defined as

$$\lim_{A\to\infty}\int_{-A}^{A},$$

the only possible values for L are $0, \pm\frac{1}{2}\pi\,,\,\pm\pi$. Since $g_o = \frac{1}{2}$, it follows by induction from (19) that for $-x \leqq \phi \leqq x$

$$|g_r(x,\phi)| \leqq 2^{2n-1}\,|x|^{2n}\,/\,n!$$

Therefore,

$$G(x,\phi,t) = \sum_{n=0}^{\infty} g_n(x,\phi)\,t^n$$

exists for all values of t. That G satisfies (5), subject to the initial conditions (6), follows from integration by parts of

$$\int_{-x}^{x} e^{2i\omega\phi} \left\{ \frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial \phi^2} + 8t \cos 2xG \right\} d\phi$$

and from equation (10) which implies that

$$\frac{d^2 u_n}{dx^2} + 4\omega^2 u_n = -8t (\cos 2x) u_{n-1} .$$

VI. Generalizations.

Most of the results of the previous sections can be generalized very easily. Let

(1) $$y'' + [4\omega^2 + 8f(x)]\, y = 0$$

be a differential equation of Hill's type. In particular, we shall assume that

(2) $$f(x) = \sum_{n=1}^{\infty} t_n \cos 2nx$$

is an even function of period π which we shall assume to be analytic everywhere. Then we can define the analogues of the infinite determinants $\Delta_0(\omega,t)$, C_+, S_+, C_-, S_- and we can prove all the inequalities in Section III if we replace t by $\sum |t_n|$. If we define the standard solutions y_1 and y_2 by the same initial conditions as before, we find exactly the same conditions under which (1) has a periodic even or odd solution of period π or 2π, and we can prove (IV.5), (IV.6) and (IV.7). We can also prove the results of section V after an appropriate modification of (V.5) and (V.6).

But a difficulty arises if we wish to generalize the equations (1) to (4) of Section IV. To do this we would have to prove that the right-hand sides in (IV.1) to (IV.4), or rather their generalizations in the present case, have simple zeros with respect to ω if the t_n are real. We would need either the analogues of the product formulas (II.24) and (II.25) for the corresponding infinite determinants, or a substitute for them. So far this difficulty has not been overcome.

VII. Application to the Construction of Transparent Media

Let $f(x)$ be a continuous function of x such that

(1) $$f(x) = k^2$$

if $x < 0$ or $x > \ell$, and

(2) $$f(x) = \sum_{m=0}^{\infty} c_n \cos 2mx$$

for $0 \leqq x \leqq \ell$, where $\ell = n\pi$, $n = 1,2,3, \ldots$. We may consider $\{f(x)\}^{\frac{1}{2}}$ as the index of refraction of a medium which is homogeneous except for a layer of thickness ℓ. A wave traveling in this medium will be defined by a function $y(x)$ which satisfies

(3) $$y'' + f(x)y = 0$$

and is continuous and has a continuous first derivative.

We shall call the medium completely transparent if for any real value of k there exists a solution $y(x)$ as defined above which satisfies

(4) $$y(x) = e^{ikx} \text{ for } x < 0,\ y(x) = e^{ik(x+\alpha)} \text{ for } x > \ell,$$

where α is real.*

Let $y_1(x)$ and $y_2(x)$ be the standard solutions of (3) in $0 \leqq x \leqq \ell$, which are defined by $y_1(0) = y_2'(0) = 1$, $y_2(0) = y_1'(0) = 0$.

Then, for $0 \leqq x \leqq \ell$

(5) $$y(x) = y_1(x) + iky_2(x)$$

because $y(x)$, $y'(x)$ are continuous for $x = 0$. For $x = \ell$ we find from (4):

(6) $$|y(\ell)| = 1, \qquad y'(\ell) = iky(\ell),$$

and therefore

(7) $$y_1^2(\ell) + k^2 y_2^2(\ell) = 1, \qquad y_1'(\ell) = iky_2(\ell),\ y_2'(\ell) = y_1(\ell).$$

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* Dr. J. Shmoys gave the following physical interpretation to this definition: If the layer is submerged in a medium of the constant index of refraction ν^2, a plane wave passes through the layer without reflection, independent of the value of k^2.

If (7) is true for all k_1 then

$$(7') \quad y_1(\ell) = \pm 1, \qquad y_2(\ell) = 0, \; y_1'(\ell) = 0, \; y_2'(\ell) = \pm 1.$$

This shows that the following is a necessary and sufficient condition for a transparent layer: The differential equation (3), in which f(x) is now given by (2) for all values of x, must have two linearly independent solutions having periods of either ℓ or 2ℓ.

The remarks in Section II show that Mathieu's equation can indeed be used for the construction of transparent layers and that this is also true in general for any equation of Hill's type (3) if we choose $\ell = 3\pi, 4\pi, 5\pi, \ldots$. We merely have to make sure that the characteristic exponent α of Section II is a rational number between 0 and $\frac{1}{2}$. This has been pointed out already by Kramers [4]. If we take $k^2 = f(o) = f(\ell)$, we obtain an index of refraction which is continuous and has a continuous first derivative everywhere. But in the cases obtained from Mathieu's equation, the inhomogeneous part of the layer consists of at least three equal sublayers. Cases where $\ell = \pi$, that is where the period of f(x) itself is also a period of all solutions of (3), have been found by Meissner [6], who used a discontinuous piecewise constant f(x), and by Klotter and Kotowski [3], whose method consists in a numerical analysis of the regions of stability for the solutions of

$$(8) \qquad y'' + (\lambda + \gamma_1 \cos x + \gamma_2 \cos 2x)\, y = 0.$$

In [3] two periodic solutions of period π are then found for $\gamma_1 = 1$, $\gamma_2 = 1/2$ and $\lambda \sim 1.17$. Finally, Bouwkamp [1] has shown that the case where f(x) represents a periodically repeated triangle can be discussed completely.

If one wishes to interpret $\{f(x)\}^{1/2}$ as the index of refraction of a transparent layer, one may require an f(x) which is positive everywhere. It is easily verified that the minima of

$$(9) \qquad f(x) = \lambda + \gamma_1 \cos x + \gamma_2 \cos 2x$$

are at the points at which either $\sin x = 0$ or $\cos x = -\gamma_1/(4\gamma_2)$. For the values of $\gamma_1, \gamma_2, \lambda$ used in [3], this gives, at the minimum, i.e., $x = 5/(6\pi)$,

(10) $$f(\tfrac{5}{6}\pi) = \lambda - \tfrac{1}{2}\gamma_1 - \tfrac{1}{2}\gamma_2 > 0$$

and at $x = \pi$

(11) $$f(\pi) = \lambda - \gamma_1 > 0.$$

Therefore, the result obtained in [3] provides such an example.

A last remark to be made in connection with transparent layers is the following one: Let

(12) $$k^2 = f(o) = f(\pi) = f(\ell)$$

and consider the phase $\emptyset(x)$ of the incoming wave $\exp(ikx)$ for $(x < 0)$ in the layer. We can define $\emptyset(x)$ by

(13) $$\emptyset(x) = k\int_0^x \frac{d\xi}{y_1^2(\xi) + k^2 y_2^2(\xi)},$$

since it can be shown that

(14) $$y_1(x) + iky_2(x) = \left\{y_1^2(x) + y_2^2(x)\right\}^{\frac{1}{2}} \exp(i\emptyset(x)).$$

(This is an elementary consequence of the fact that y_1 and y_2 satisfy a homogeneous linear differential equation of second order which does not have a term involving y'.) The increase in phase of the wave at the end of the layer will be

(15) $\emptyset(\ ) = \emptyset(\pi) = 2n\pi$, $\qquad (n = 1,2,3, \ldots)$

since the left-hand side in (14) must be unity for $x = \pi$ according to (7'). It is easily seen that the integer n does not depend on k, because the indefinite integral on the right-hand side in (13) can be written as

$$\tan^{-1}\left(\frac{y_1(x)}{ky_2(x)}\right).$$

Therefore, $\emptyset(\pi)$ depends only on the number of zeros of $y_2(x)$ in the interval $0 \leq x \leq \pi$. Without the layer, the phase of the incoming wave at $x = \pi$ would have been $k\pi$. If it would be possible to choose $k = \{f(o)\}^{1/2}$ in such a way that $k = 2n$, where n is the same as in (15), we would have a layer which would not even cause a loss of phase. So far a construction of such a layer has not been carried out, but there does not appear to be any reason why it could not

exist. Of course, the trivial case where $f(x) \equiv k^2$ should not be considered.

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